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SOME REMARKS ON PROFINITE HNN EXTENSIONS

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ABSTRACT

We extend a construction of Higman, Neumann and Neumann [LS, IV.3.1] and show that every profinite group G with only countably many open subgroups embeds in a 2-generated profinite group E in which all torsion elements are conjugate to elements of G; if G is pro-p, E can be chosen pro-p. This answers a question of Wilson (oral communication) and generalises a result of Lubotzky and Wilson [LW].

Introduction

Let G be a profinite group and f: $A \to B$ a continuous isomorphism between the closed subgroups A and B of G. The definition of HNN extension extends to the category of profinite groups with continuous morphisms as follows [ZM1, (3.3)]): let H denote the usual (discrete) HNN extension $G*_f\langle t\rangle = \langle G, t|\, t^{-1}at = f(a),\ a \in A\rangle$ and let $\mathcal{S}$ denote the family of normal subgroups N of finite index in H and such that $N \cap G$ is open in G. Then

$$\hat{H} = \varprojlim_{N\in\mathcal{S}} H/N$$

is called the profinite HNN extension associated to G and f; if i: $G \to \hat{H}$ is the continuous homomorphism induced by the inclusion of G in H, then $\hat{H}$ and i have the usual universal property: if j: $G \to E$ is a continuous morphism of profinite groups, and if in E there is an element s such that $s^{-1}j(a)s = j\circ f(a)$ for every $a \in A$, then there is a unique continuous morphism π: $\hat{H} \to E$ which sends t to s and is such that j and $\pi\circ i$ agree on G.

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If G is pro-p, then the maximal pro-p-quotient $\hat{H}(p)$ of $\hat{H}$, together with the induced morphism i_p: $G \to \hat{H}(p)$ is called the pro-p-HNN extension associated to G and f.

Unlike the discrete case however, the homorphisms i and i_p are not necessarily injective. In section 1, we study what their kernels are, and under which conditions they are injective. It turns out that the obvious necessary conditions are also sufficient.

These results are of independent interest: HNN extensions and amalgamated products are central in the Bass-Serre theory of profinite groups developed by Gildenhuys and Ribes [GR], Mel'nikov and Zalesskii [ZM1, ZM2], and we show how our description of the kernels, together with results of Ribes [R] on amalgamated products, can be used to describe the kernels in finite graph products of profinite groups.

In section 2 we construct the group E.

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1. Description of the kernels

THEOREM 1.1: *With the notation as above,* $\ker i = K$, *where*

$$K = \bigcap \{U \mid U \text{ open normal in } G,\ f(A \cap U) = B \cap U\}.$$

Thus, i is injective if and only if

(∗) *For every open normal subgroup U of G there is an open normal subgroup V of G contained in U and such that $f(A \cap V) = f(A) \cap V$.*

In this case, the group H is residually finite.

Proof: From the definition of $\hat{H}$, it follows that $\ker i = \bigcap \{N \cap G \mid N \in \mathcal{S}\}$. Let $N \in \mathcal{S}$. Then $N \cap G$ is an open normal subgroup of G; from $f(A \cap N) = (A \cap N)^t = A^t \cap N = B \cap N$ we obtain $\ker i \subset K$.

Conversely, let U be an open normal subgroup of G such that $f(A \cap U) = B \cap U$. The isomorphisms $A/A \cap U \simeq AU/U$ and $B/B \cap U \simeq BU/U$ induce an isomorphism $\bar{f}$: $AU/U \to BU/U$. The canonical epimorphism $H \to (G/U) *_{\bar{f}} \langle t \rangle$ induces then a continuous epimorphism from $\hat{H}$ onto the profinite HNN extension associated to G/U and $\bar{f}$. The second inclusion follows then from the residual finiteness of $(G/U) *_{\bar{f}} \langle t \rangle$ [BT, Theorem 3.1].

Let X and Y be sets of representatives for G/A, G/B respectively, with 1 in their intersection. Then every element g of H is written in a unique way (called the normal form) as

$$g = g_1 t^{\epsilon_1} g_2 t^{\epsilon_2} \dots g_n a$$

where $\epsilon_i \in \{1, -1\}$, $\epsilon_i = 1$ implies $g_i \in X$, $\epsilon_i = -1$ implies $g_i \in Y$, $g_n \in X$, $a \in A$, and no occurrence of $t^{-1}1t$ or $t1t^{-1}$ appears. Since A and B are closed, there is an open normal subgroup U of G such that $a \neq 1$ implies $a \notin U$, $g_i \notin A$ implies $g_i \notin AU$ and $g_i \notin B$ implies $g_i \notin BU$. By $(*)$, we can assume that $f(A \cap U) = B \cap U$. If p is the canonical epimorphism from H onto $(G/U) *_{\bar{f}} \langle t \rangle$, then $p(g)$ is written in reduced form as $p(g_1)t^{\epsilon_1}p(g_2)t^{\epsilon_2} \dots p(g_n)p(a)$ and is therefore non-trivial by Britton's Lemma. The residual finiteness of H now follows from the residual finiteness of $(G/U) *_{\bar{f}} \langle t \rangle$.

Clearly, the injectivity of i_p implies the injectivity of i, and thus condition $(*)$ must hold for i_p to be injective. It turns out however that this condition is not sufficient.

Indeed, consider the group of exponent p generated by two elements a and b such that $[a,b]$ is central. Let $A = \langle a, [a,b] \rangle$ and let f be defined by $f(a) = [a,b]$, $f([a,b]) = b$. But in a finite p-group, the equation $[a,b] = a^x$ implies $a = 1$! Indeed, if $\gamma_i(G)$ denotes the i-term of the lower central series of G, then $a \in \gamma_i(G)$ implies $a^x \in \gamma_i(G)$ and $[a,b] \in \gamma_{i+1}(G)$. Thus the pro-$p$ HNN extension associated to G and f is the group $\langle t \rangle$. ∎

LEMMA 1.2: *Let G be a finite p-group, $f: A \to B$ an isomorphism between two subgroups of G. Suppose that G has a chief series $(C_i)_{i \leq n}$ satisfying*

$(**)$ $f(A \cap C_i) = f(A) \cap C_i$ and f induces the identity on $(AC_i \cap C_{i-1})/C_i$ for every $i > 0$.

Then G embeds in a finite p-group T in which f is induced by conjugation and $(C_i)_{i \leq n}$ refines to a chief series.

Proof: We consider first the case where the restriction of f to $A \cap C_1$ is the identity. Let $U = C_1$. If $A \subset U$, there is nothing to prove. Suppose therefore that there is an element a in $A \setminus U$ and let $b = f(a)$. Then $ab^{-1} \in U$. From the fact that f is the identity on $A \cap U$, we deduce that $a^p = b^p$ and $g^a = g^b$ for every $g \in A \cap U$.

Let C be the cyclic group of order p generated by the element $\bar{a}$, and consider the wreath product $U \wr C$. Recall that it is defined as the semi-direct product of U^C by C, where the action of C on U^C is given by $s^g(h) = s(gh)$ for $s \in U^C$, g and h in C, and the multiplication is given by $(s_1, h_1)(s_2, h_2) = (s_1^{h_2} s_2, h_1 h_2)$ for s_1, s_2 in U^C and h_1, h_2 in C.

One defines an embedding of G into $U \wr C$ as follows: let π: $G \to C$ be the epimorphism sending U to 1 and a to $\bar{a}$, and let θ: $G \to G$ be defined by $\theta(a^i u) = a^i$ for $0 \le i < p$ and $u \in U$. For g in G, one then defines $s_g \in U^C$ by: $s_g(\bar{a}^i) = [\theta(ga^i)]^{-1} g a^i$.

It is then well-known that the map $g \mapsto (s_g, \pi(g))$ defines an embedding of G into $U \wr C$. Let $s \in U^C$ be defined by

$$s(\bar{a}^i) = b^{-i} a^i.$$

We claim that $(s, 1)$ is our desired element. By definition, we have

$$s_a(\bar{a}^i) = \begin{cases} 1 & \text{if } i < p-1, \\ a^p & \text{if } i = p-1, \end{cases}$$

and

$$s_b(\bar{a}^i) = \begin{cases} a^{-i-1} b a^i & \text{if } i < p-1, \\ b a^{p-1} & \text{if } i = p-1. \end{cases}$$

Also $(s,1)^{-1}(s_a, \bar{a})(s,1) = ((s^{-1})^{\bar{a}} s_a s, \bar{a})$ and

$$\begin{aligned} (s^{-1})^{\bar{a}} s_a s(\bar{a}^i) &= (s^{-1})^{\bar{a}}(\bar{a}^i) s_a(\bar{a}^i) s(\bar{a}^i) \\ &= \begin{cases} a^{-i-1} b a^i & \text{is } i < p-1, \\ a^p b^{-p+1} a^{p-1} & \text{is } i = p-1, \end{cases} \\ &= s_b(\bar{a}^i) \end{aligned}$$

because $a^p b^{-p} = 1$. Thus $(s,1)^{-1}(s_a, \pi(a))(s,1) = (s_b, \pi(b))$. Let $u \in A \cap U$. Then $s_u(\bar{a}^i) = a^{-i} u a^i$ belongs to $A \cap U$. Hence $s^{-1} s_u s = s_u$. This shows that in $U \wr C$, conjugation by s coincides with f on A. Clearly (C_i) refines to a chief series of $U \wr C$.

We will show by induction on the size of G that we can reduce the general case to the first case.

Let $(C_i)_{i \le n}$ be a chief series satisfying (**). If G is trivial, there is nothing to prove. By induction hypothesis, C_1 embeds in a finite p-group T_1 in which the restriction of f to $A \cap C_1$ is induced by conjugation by an element s and $(C_i)_{1 \le i \le n}$

refines to a chief series of T_1. By Higman's amalgamation theorem [H], the (discrete) amalgamated product $T_1 *_{C_1} G$ is residually p and has a finite p-quotient $T_2 = (T_1 *_{C_1} G)/N$ such that T_1 and G embed in T_2, and $(C_i)_{1 \leq i \leq n}$ refines to a chief series of T_2. Observe that these properties are preserved by inverse image. Intersecting if necessary N with the normal subgroup of $T_1 *_{C_1} G$ generated by T_1, we may therefore assume that T_2 has a maximal normal subgroup U which contains s and intersects G in C_1.

Thus, considering the partial isomorphism g defined by $g(a) = sf(a)s^{-1}$ for $a \in A$ allows us to use the first case. ∎

THEOREM 1.3: *Let G be a pro-p-group and $f\colon A \to B$ an isomorphism between two closed subgroups of G. Let $\mathcal{T}$ be the set of open normal subgroups U of G such that there are normal subgroups $U = C_n \subset C_{n-1} \subset \cdots \subset C_0 = G$ satisfying*

(**) *$f(A \cap C_i) = B \cap C_i$ and f induces the identity on $(AC_i \cap C_{i-1})/C_i$ for every $i > 0$.*

Then $\ker i_p = \bigcap\{U \mid U \in \mathcal{T}\}$. Thus i_p is injective if and only if every open subgroup of G contains an element of $\mathcal{T}$. In this case H is residually p.

Proof: Let $\mathcal{S}_p$ be the set of elements of $\mathcal{S}$ which have index a power of p. Then $\hat{H}(p)$ is the inverse limit of the quotients H/N where N ranges over $\mathcal{S}_p$. Thus $\ker i_p = \bigcap\{N \cap G \mid N \in \mathcal{S}_p\}$.

Let $N \in \mathcal{S}_p$, and let $(D_i)_{i \leq m}$ be a chief series for H/N; the distinct members of $\{D_i \cap G \mid i \leq m\}$ then give a chief series for $G/G \cap N$ which satisfies (**). Thus $N \cap G \in \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$ and (C_i) the associated sequence of normal subgroups. Since f induces the identity on each $(AC_i \cap C_{i-1})/C_i$, the sequence (C_i) refines to a chief series (C_i') satisfying (**). Let $\bar{f}\colon AU/U \to BU/U$ is the isomorphism induced by f; by the lemma, $(G/U) *_{\bar{f}} \langle t \rangle$ has a normal subgroup of index a power of p which intersects G/U trivially. The inverse image of this normal subgroup in H is a member of $\mathcal{S}_p$ which intersects G in U.

The proof that if i_p is injective then H is residually p is done as in Theorem 1.1. ∎

Remark: Let $(G_i)_{i \geq 0}$ be the decreasing chain of closed subgroups of G defined as follows: $G_0 = G$, G_{i+1} is the closed subgroup generated by $G_i^p[G, G_i]$ and $\{f(a)a^{-1} \mid a \in A \cap G_i\} \cup \{f^{-1}(a)a^{-1} \mid a \in B \cap G_i\}$. Then each G_i is normal in G and they satisfy (**); if G is finitely generated, they are open and $\ker i_p = \bigcap G_i$.

APPLICATION TO GRAPH PRODUCTS OF PROFINITE GROUPS. Recall from [ZM1, (3.1)] that a finite graph of profinite group is given by

— a finite graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, together with two maps d_0 and d_1 from $E(\Gamma)$ to $V(\Gamma)$.

— for each vertex v a profinite group $\mathcal{G}(v)$; for each edge e a profinite group $\mathcal{G}(e)$ and embeddings α_0^e: $\mathcal{G}(e) \to \mathcal{G}(d_0(e))$ and α_1^e: $\mathcal{G}(e) \to \mathcal{G}(d_1(e))$. Let us denote by A_e and B_e the groups $\alpha_0^e(\mathcal{G}(e))$ and $\alpha_1^e(\mathcal{G}(e))$.

The graph product is then defined as in the discrete case: for each edge e, let f_e be the isomorphism $\alpha_1^e(\alpha_0^e)^{-1}$: $A_e \to B_e$; let T be a maximal subtree of Γ, and let S be the (profinite or pro-p) tree product of the profinite groups $\mathcal{G}(e)$, $e \in E(\Gamma)$, with amalgamated subgroups A_e and B_e for $e \in T$ (via f_e); the (profinite or pro-p) graph product is then the (profinite or pro-p) HNN extension Π of S with respect to the partial isomorphisms f_e for $e \in E(\Gamma) \setminus T$ (on the letters t_e). We also have natural maps ϕ_v: $\mathcal{G}(v) \to \Pi$ and ϕ_e: $\mathcal{G}(e) \to \Pi$. An easy application of 1.1 and 1.2 and of Theorem 1.2 in [R] gives then the following:

Profinite graph product: Let U be an open normal subgroup of $\mathcal{G}(v)$ for some vertex v; then U contains $\ker \phi_v$ if and only if there are open normal subgroups $V(w)$ of $\mathcal{G}(w)$, $w \in V(\Gamma)$, such that $V(v) \subset U$ and for each edge e, $f_e(A_e \cap V(d_0(e))) = B_e \cap V(d_1(e))$.

Pro-p graph product: Let U be an open normal subgroup of $\mathcal{G}(v)$ for some vertex v; then U contains $\ker \phi_v$ if and only if for each vertex w there is a decreasing chain $(V(w,i))_{i \leq n}$ of open normal subgroups of $\mathcal{G}(w)$ such that $V(v,n) \subset U$, $f_e(A_e \cap V(d_0(e),i)) = B_e \cap V(d_1(e),i)$ and f_e induces the identity on each $(A_e V(d_0(e),i) \cap V(d_0(e),i-1))/V(d_0(e),i)$ for every edge e and $i \leq n$.

2. The construction

We now fix a profinite group G with countably many open subgroups, and will construct a 2-generated profinite group E in which G embeds and where all the torsion elements are conjugate to elements of G; if G is pro-p, then so will E.

We will do both constructions at the same time. Let F be the free profinite [resp pro-p] group on $a_1, \ldots, a_p$ and let σ be the automorphism of F defined by $a_1 \mapsto a_2$, $\ldots$, $a_{p-1} \mapsto a_p$, $a_p \mapsto a_1$. Let N be the closed normal subgroup of F generated by $a_1, \ldots, a_{p-1}$. Then N is free on countably many generators [M, Proposition 4.1]. Let $X = \{a_p^{-i} a_j a_p^i \mid i \in \mathbf{Z},\ 1 \leq j \leq p-1\}$; the discrete subgroup of F generated by X is dense in N; since $a_1, \dots, a_{p-1} \in X$, there is a set of topological generators of N containing them and therefore we can choose an epimorphism $\pi: N \to G$ with $a_1, \dots, a_{p-1}$ in its kernel.

Consider the subgroup $A = N \times (1)$ of $F \times G$ and the isomorphism $f: A \to B = f(A)$ defined by: $f(a, 1) = (\sigma(a), \pi(a))$. Then f is clearly continuous, and we now consider the profinite [resp., pro-p] HNN extension E associated to $F \times G$ and f.

We will first show that $F \times G$ embeds into E, i.e., that $F \times G$ and f satisfy $(*)$ [or $(**)$].

Let U be an open normal subgroup of $F \times G$. Then U contains an open normal subgroup of the form $U_1 \times U_2$; since π is continuous we may assume that $U_1 \subset \pi^{-1}(U_2)$; because σ has order p, we may assume that $\sigma(U_1) = U_1$. Then

$$f(A \cap (U_1 \times U_2)) = \{(\sigma(a), \pi(a)) |\, a \in N \cap U_1\}$$

and

$$B \cap (U_1 \times U_2) = \{(\sigma(a), \pi(a)) |\, a \in N \cap \sigma^{-1}(U_1) \cap \pi^{-1}(U_2)\}.$$

Thus $f(A \cap (U_1 \times U_2)) = B \cap (U_1 \times U_2)$, which shows $(*)$.

For $(**)$, observe first that $F \rtimes \langle \sigma \rangle$ is a pro-p-group and that U_1 is a normal subgroup. Let $F = C_1 \supset C_2 \supset \cdots \supset C_n = U_1$ be a chain of normal subgroups of $F \rtimes \langle \sigma \rangle$ with $[C_i : C_{i+1}] = p$; then σ induces the identity on C_i / C_{i+1}, $\pi(C_i)$ is a normal subgroup of G and $[\pi(C_i) : \pi(C_{i+1})]$ equals 1 or p.

We will refine the series $C_i \times \pi(C_i)$ of $F \times G$ to one satisfying $(**)$; this is done by induction on i; suppose that $(D_j)_{j \leq m}$ has already been constructed satisfying $(**)$, with $D_m = C_i \times \pi(C_i)$. There are two cases to consider:

Case 1: $\pi(C_i) = \pi(C_{i+1})$. Let $D_{m+1} = C_{i+1} \times \pi(C_i)$. Then

$$f(D_{m+1} \cap A) = \{(\sigma(a), \pi(a)) |\, a \in N \cap C_{i+1}\}$$

and

$$D_{m+1} \cap B = \{(\sigma(a), \pi(a)) |\, a \in N \cap \sigma^{-1}(C_{i+1}) \cap \pi^{-1}(\pi(C_i))\},$$

which implies that $f(D_{m+1} \cap A) = D_{m+1} \cap B$. If $a \in C_i \cap N$, then $a\sigma(a)^{-1} \in C_{i+1}$, and therefore $(a, 1) f(a, 1)^{-1} \in D_{m+1}$, which shows $(**)$.

CASE 2: $\pi(C_i) \neq \pi(C_{i+1})$. Let $D_{m+1} = C_{i+1} \times \pi(C_i)$, and $D_{m+2} = C_{i+1} \times \pi(C_{i+1})$. As above, one shows that $f(D_{m+1} \cap A) = D_{m+1} \cap B$ and $f(D_{m+2} \cap A) = D_{m+2} \cap B$. If $a \in C_i \cap N$, then $a\sigma(a)^{-1} \in C_{i+1}$ and therefore $(a,1)f(a,1)^{-1} \in D_{m+1}$; since $D_{m+1} \cap A = D_{m+2} \cap A$, we obtain $(**)$.

This finishes the proof that $F \times G$ embeds inside E. By of [ZM1, (3.10)] (see also (3.2)), any torsion element of E is conjugate to a torsion element of $F \times G$, i.e., of G. We now claim that E is generated by $(a_1, 1)$ and t: indeed, conjugating $(a_1, 1)$ by t, we obtain successively $(a_2, 1), \ldots, (a_p, 1)$ and therefore $F \times (1) \subset \langle (a_1, 1), t \rangle$; this in turn implies that $B \subset \langle (a_1, 1), t \rangle$, and because $F \times G = \langle F \times (1), B \rangle$, finishes the proof of the claim.

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